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Correlation of eigenstates in the critical regime of quantum Hall systems

Krystian Pracz, Martin Janssen and Peter Freche
Institut für Theoretische Physik, Universität zu Köln, Zülpicher Strasse 77, 50937 Köln, Germany

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Abstract. We extend the multifractal analysis of the statistics of critical wave functions in quantum Hall systems by calculating numerically the correlations of local amplitudes corresponding to eigenstates at two different energies. Our results confirm multifractal scaling relations which are different from those occurring in conventional critical phenomena. The critical exponent corresponding to the typical amplitude, $\alpha_0 \approx 2.28$, gives an almost complete characterization of the critical behaviour of eigenstates, including correlations. Our results support the interpretation of the local density of states being an order parameter of the Anderson transition.

1. Introduction

Two-dimensional independent electrons in the presence of static disorder and a strong magnetic field undergo disorder induced localization–delocalization (LD) transitions when the Fermi energy crosses critical energies—the Landau energies. These LD transitions are believed to be responsible for the integer quantum Hall effect [1, 2]. In finite-size systems the localization length ξ of the electronic states is larger than the system size L for a certain energy range, ΔE, around the critical energies. These states are called *critical states*. In the thermodynamic limit $\Delta E \propto L^{-1/\nu}$ where ν is the critical exponent of ξ. An obvious task for a theory of the LD transition is to yield the statistics and scaling behaviour of eigenstates in the critical regime. After pioneering works by Wegner [3] and Aoki [4] it became clear that the critical wave functions have a multifractal structure (for a review see [5] and references therein). The entire distribution of local amplitudes and its scaling behaviour is encoded in the multifractal $f(\alpha)$ spectrum. The distribution is broad on all length scales and close to a log-normal distribution. The most important quantity is the maximum position, α_0, of $f(\alpha)$. It describes the scaling behaviour of the geometric mean of what serves as a *typical* amplitude of a critical wave function. The $f(\alpha)$ spectrum has been interpreted as a spectrum of critical exponents related to the order parameter field of the LD transition [5]. To be consistent with such an interpretation $f(\alpha)$ has to share several features with scaling exponents in conventional critical phenomena. Firstly, it has to be universal, i.e. independent of the disorder configuration and the microscopic details of the electron state. In previous studies this was confirmed [6]. Secondly, correlations of the order parameter field have to be related to $f(\alpha)$ by appropriate scaling relations. A systematic investigation of this topic is presented here (related conformal scaling relations were recently investigated by Dohmen *et al* [7]). Our numerical data are consistent with universal scaling relations between $f(\alpha)$ and scaling exponents of the energetic and spatial correlations of critical wave functions.

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The article is organized as follows. In section 2 we explicitly demonstrate that the multifractal spectrum describes the distribution function of local amplitudes. In section 3 the spatial correlations of local amplitudes (and powers thereof) of wave functions at a fixed energy in the critical regime are investigated. We study the scaling with respect to the size of boxes which the local amplitude is averaged over, and with respect to the distance between correlated amplitudes. We find that scaling relations are fulfilled which can also be obtained by heuristic arguments. In section 4 we study the correlator of two local amplitudes corresponding to critical eigenstates at energies separated by ω. We find that a characteristic length scale L_ω (introduced by Chalker [8, 9]) serves as a cutoff length for multifractal correlations. Remarkably, two local amplitudes corresponding to eigenstates separated in energy by ω are correlated in the same way as two amplitudes corresponding to one eigenstate, provided their spatial distance is less than L_ω. By transforming from the wave function amplitudes to the local density of states our results show that the local density of states has several features in common with order parameter fields. The conclusions are summarized in section 5.

2. Distribution of local amplitudes

In this section we first review the multifractal analysis of the distribution of local amplitudes of critical eigenstates. Secondly, we confirm that the distribution is encoded in the multifractal $f(\alpha)$ spectrum by a direct comparison of the numerically obtained histogram of local amplitudes with the theoretical distribution function that follows from the numerically obtained $f(\alpha)$ spectrum.

In 1983 Aoki [4] gave a nice argument for the multifractal behaviour of critical wave functions (although at that time the phrase 'multifractality' was not yet common). His argument goes as follows. Consider the inverse participation number defined by

$$\mathcal{P} = \int_\Omega \mathrm{d}^d r\, |\psi(\boldsymbol{r})|^4 \tag{1}$$

where Ω denotes a d-dimensional region with linear size L. If the wave function $\psi(\boldsymbol{r})$ is uniformly distributed—as in a metallic phase—then $\mathcal{P} \propto L^{-d}$ and the participation ratio $p = (\mathcal{P}L^d)^{-1}$ is constant. In the localized regime $\mathcal{P} \approx \xi^{-d}$ and p vanishes in the thermodynamic limit. At the transition point where the wave function is extended the participation ratio still has to vanish in the thermodynamic limit if the LD phenomenon is similar to a second-order phase transition. Consequently, $\mathcal{P}$ scales with a power $d^* < d$. Wegner had already calculated the whole spectrum of exponents for generalized inverse participation numbers within the non-linear sigma model [3]. This spectrum was interpreted as a multifractal spectrum by Castellani and Peliti [10]. After extensive numerical work the following description of the statistics of critical wave functions $\psi(\boldsymbol{r})$ is now established.

Consider the box probability

$$P(L_b) := \int_{\mathrm{box}} |\psi(\boldsymbol{r})|^2 \tag{2}$$

of some box with linear size L_b, normalized to the total volume L^d, $P(L) = 1$. At the LD transition the corresponding distribution function $\pi(P; L_b/L)$ gives rise to power law scaling for the moments,

$$\langle [P(L_b)]^q \rangle_L \propto (L_b/L)^{d+\tau(q)} \tag{3}$$

where $d + \tau(q)$ is a non-linear function of q. This non-linearity is a direct consequence of Aoki's observation that $d + \tau(2) = d + d^* \neq d + d$. The brackets $\langle \ldots \rangle_L$ in (3)

denote the average over disorder configurations. In practice it turns out that, to a good accuracy, this average can often be substituted by the spatial average over one wave function for a given configuration. Within numerical accuracy the resulting spectra are identical. This 'universality' is expected to be precise in the thermodynamic limit (see also [11]).

The corresponding (universal) distribution function can be described in terms of a single-humped, positive function $f(\alpha)$ called the multifractal spectrum of the wave function,

$$\pi(P; L_b/L)\,\mathrm{d}P \propto (L_b/L)^{d-f(\alpha)}\,\mathrm{d}\alpha \tag{4}$$

where $\alpha := \ln P/\ln(L_b/L)$; $f(\alpha)$ is related to $\tau(q)$ by a Legendre transformation

$$f(\alpha(q)) = \alpha(q)q - \tau(q) \qquad \alpha(q) = \mathrm{d}\tau(q)/\mathrm{d}q. \tag{5}$$

Thus, the statistics of critical eigenstates is encoded in $f(\alpha)$ or equivalently in $\tau(q)$. Because of the horse-shoe shape of $f(\alpha)$ it can be approximated by a parabola,

$$f(\alpha) \approx d - (\alpha - \alpha_0)^2/(4(\alpha_0 - d)). \tag{6}$$

This parabolic approximation (PA) contains α_0 as the only parameter besides d. This is due to the assumed validity of the PA at least up to $|q| \leqslant 1$. Equation (6) corresponds to a log-normal distribution centred around the typical value $P_{\mathrm{typ}} := \exp\langle \ln P\rangle \propto (L_b/L)^{\alpha_0}$ with log variance proportional to $\alpha_0 - d$. A simple one-parameter approximation for $f(\alpha)$, which takes into account that the support $[\alpha(\infty), \alpha(-\infty)]$ of $f(\alpha)$ is finite, is the semi-elliptic approximation (SEA)

$$f(\alpha) \approx d\sqrt{1 - \frac{(\alpha - \alpha_0)^2}{\alpha_0^2 - d^2}}. \tag{7}$$

To demonstrate that the distribution of local amplitudes of critical eigenstates is encoded in $f(\alpha)$ we present numerical results for a quantum Hall system (QHS).

The wave functions are calculated for the model of independent (spinless) electrons subject to strong magnetic field and disorder. The disorder was implemented by a set of δ-impurities with random positions and random strengths symmetric around zero. To avoid some of the (degenerate) wave functions of the pure system having zeros precisely at the position of the point scatterers (and thus not being affected by disorder) the number of point scatterers was taken to be larger then the number of flux quanta penetrating the system's area L^2 [12]. The microscopic length scale of the problem is the magnetic length l_B defined by the size of a cell penetrated by a single flux quantum, $2\pi l_B^2$. We worked out the representing Hamiltonian matrix in the Landau representation which is convenient for periodic boundary conditions in one direction (say the y-direction). To account for a finite system size in the x-direction we adopted the Landau counting procedure which results in a matrix of dimension $N = L^2/(2\pi l_B^2)$, after projecting to the lowest Landau band. The matrix has band structure with a bandwidth of order $\sqrt{N}$ and allows for diagonalizing systems of linear size L about $200 l_B$ with the aid of usual workstations. The diagonalization yields the eigenvalues and eigenstates for any desired energy window within the lowest Landau band. The determination of the range ΔE of critical states was based on a previous analysis of Thouless numbers within the same model [13].

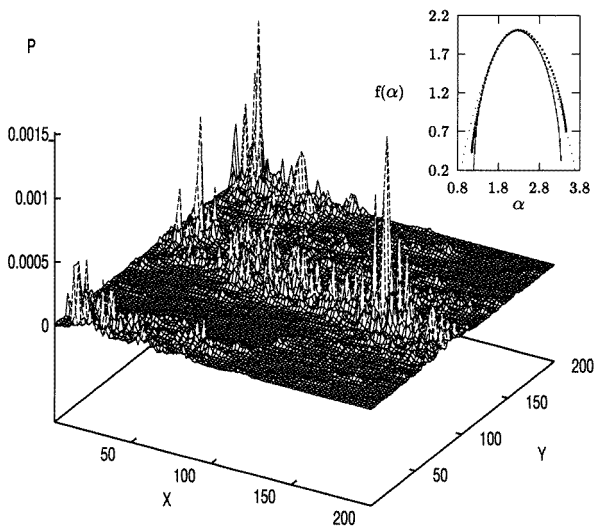

Figure 1. Squared amplitudes of a critical wave function for a system of linear size 200 magnetic lengths. The corresponding $f(\alpha)$ spectrum (●) together with the parabolic approximation (·····) (PA, equation (6)) and the semi-elliptic approximation (——) (SEA, equation (7)) are also shown.

In figure 1 the squared amplitudes of a wave function from the centre of the Landau band are shown together with the $f(\alpha)$ spectrum calculated from these amplitudes. The corresponding histogram of the logarithm of amplitudes (measured on a box of size $4l_B^2$) is displayed in figure 2 together with the distribution function calculated from the $f(\alpha)$ spectrum using (4). These figures demonstrate that the distribution of amplitudes (i) is encoded in the $f(\alpha)$ spectrum and (ii) is close to a log-normal distribution characterized by one critical exponent $\alpha_0 = 2.28 \pm 0.03$ (the average over 130 critical states). Similar findings have been presented already in section 12.2 of [2].

3. Correlations at fixed energy

In this section we first outline a theoretical description of scaling relations between the critical exponents of the correlations at a fixed energy and the $f(\alpha)$ spectrum. We follow mainly the presentation of [5]. Also, we present our numerical data which confirm the scaling relations.

To study the spatial correlations of amplitudes for a fixed energy we consider the q-dependent correlations

$$M^{[q]}(r, L_b, L) := \langle [P_i(L_b)]^q [P_{i+s}(L_b)]^q \rangle_L \tag{8}$$

where the average is to be taken over all pairs of boxes with fixed distance $r = sL_b$.

For critical states where the microscopic scale (l_B in our case) and the macroscopic scale (the localization length ξ in our case) are separated, one can expect power law behaviour of $M^{[q]}$ in the regime

$$l_B \ll L_b, r, L \ll \xi. \tag{9}$$

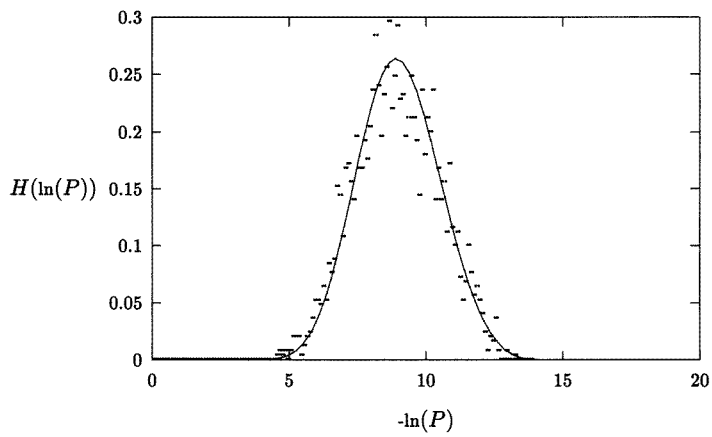

Figure 2. Histogram of the logarithm of squared amplitudes shown in figure 1. The continuous curve is the distribution function following from the corresponding $f(\alpha)$ spectrum via equation (4).

Usually, in critical phenomena one studies correlations for infinite system size (and L_b being microscopic) as a function of r alone. This is justified if, for large enough system sizes L, the correlation function is independent of L (for simplicity we neglect any trivial L-dependence due to prefactors in the definition of the observable P). However, this is not true in the multifractal case. Multifractality reflects broadness of the distribution function $\pi(P, L_b/L)$ on all length scales. This is due to the dependence of the box probability in each box on a large number of conditions, simultaneously. More generally, the local box observable $P_i(L_b)$ depends on a large number of conditions for the entire system of linear size L, simultaneously. This behaviour was denoted as 'many-parameter (MP) coherence' [5]. In the context of the LD transition, coherence at zero temperatures is due to quantum mechanical phase coherence of the electron's wave function, and disorder introduces a huge number of parameters, e.g. the position of point scatterers. In the case of MP coherence one has to face the fact that $M^{[q]}$ depends non-trivially on L, even for $L \to \infty$. To distinguish between the multifractal and the 'ordinary' situations one can implicitly define a length scale $\hat{L}$ by the requirement that $M^{[q]}$ will be independent of L for $L > \hat{L}$. We call $\hat{L}$ the MP coherence length. Still, two different cases have to be distinguished. It may happen that $\hat{L}$ introduces a cutoff for correlations. For example, the correlation length ξ is an MP coherence length of this kind. Alternatively, $\hat{L}$ does not introduce a cutoff and correlations still show a modified power law behaviour with respect to r for $r \gg \hat{L}$. Such a kind of MP coherence length occurs in ordinary critical behaviour where $\hat{L}$ is microscopic and no multifractality occurs, i.e. $\tau(q) = d(q-1)$.

After these general considerations we can formulate our expectations for the present case of correlations of critical eigenstates at fixed energy. Because of the multifractal character of the critical states whose range is only limited by the localization length $\xi \gg L$ we can expect to be dealing with an MP coherent situation with L being the MP coherence length $\hat{L}$, setting the cutoff for correlations. Therefore, we consider the regime $l_B \ll L_b < r < L \ll \xi$ and make the *ansatz*

$$M^{[q]}(r, L_b, L) \propto L_b^{x_2(q)} L^{-y_2(q)} r^{-z(q)}. \tag{10}$$

The task is now to find the scaling relations between the set of exponents

$x_2(q), y_2(q), z(q)$ and the $\tau(q)$ function of equation (3). These scaling relations can be derived on the basis of heuristic arguments. Consider the limiting situations (i) where r is of the order of L_b and (ii) where r is of the order of L. In case (i) the function $M^{[q]}$ will behave like $\langle [P(L_b)]^{2q}\rangle_L$ while in case (ii) a decoupled behaviour occurs, $M^{[q]} \sim (\langle [P(L_b)]^q\rangle_L)^2$. The uniqueness of scaling exponents allows then to conclude the desired scaling relations

$$y_2(q) = d + \tau(2q) \tag{11}$$

$$x_2(q) = 2d + 2\tau(q) \tag{12}$$

$$z(q) = d + 2\tau(q) - \tau(2q) \tag{13}$$

already proposed in [14, 5]. It is worth mentioning that the sum $x_2(q) - y_2(q) - z(q)$ vanishes due to the normalization of the wave function. To verify numerically the scaling relations it is thus enough to verify two of the three equations given above. Therefore, we concentrate on the exponents $z(q)$ and $x_2(q)$ for which we can set a fixed system size in numerical calculations. This reduces the computational effort substantially.

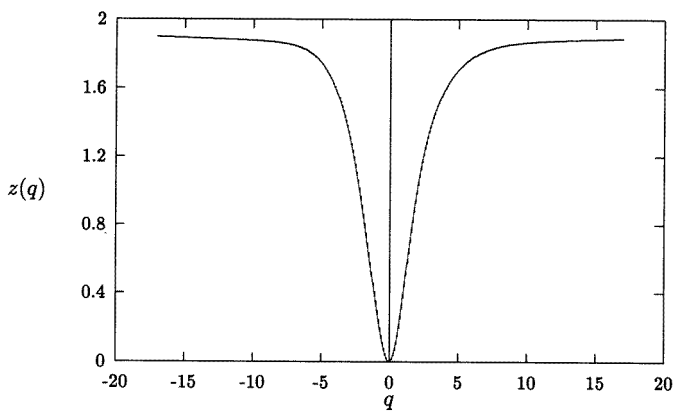

Figure 3. The function of critical exponents $z(q)$ following from the scaling relation equation (13).

Let us summarize the analytic behaviour of $x_2(q)$ and $z(q)$ according to equations (12) and (13): $x_2(q)$ is a monotonic increasing function with negative curvature and asymptotic slopes given by $2\alpha(\mp\infty)$. It vanishes at $q = 0$. $z(q)$ is non-negative with minimum at $(0, 0)$. For $q > 0$ (< 0) it is monotonically increasing (decreasing) and is asymptotically bounded by the dimension d (see figure 3). To check on the validity of equation (13) we took 100 critical states of a system with $L = 200l_B$ and calculated $M^{[q]}(L_b, r, L)$ with fixed values $L_b = l_B, 4l_B$; $L = 200l_B$. The distance r was varied from L_b to $150l_B$. The periodic boundary conditions in the y-direction reduce the upper scale for a reliable determination of exponents to $r \lesssim 100l_B$. As can be seen from figure 4 the power law behaviour holds up to $\approx 60l_B$. The numerical data for $z(q)$ were obtained by determining the slope in the linear regime of the plots of $\ln M^{[q]}$ against $\ln r$. In figure 5 the average of $z(q)$ data over 100 states is shown in the regime $|q| < 2$. For comparison the function $d + 2\tau(q) - \tau(2q)$ is plotted, too. Within the errors the validity of the scaling relation equation (13) can be confirmed. For later comparison we mention that $z(1) = 0.43 \pm 0.05$, $z(0.5) = 0.13 \pm 0.03$ and $\mathrm{d}^2z/\mathrm{d}q^2(q = 0) = 1.1 \pm 0.08 \approx 4(\alpha_0 - d)$. To determine the exponents $x_2(q)$ numerically we fixed $r = 100l_B$, $L = 200l_B$ and varied L_b from $2l_B$ to r. As shown in figure 6 the scaling relation equation (12) is fulfilled, too.

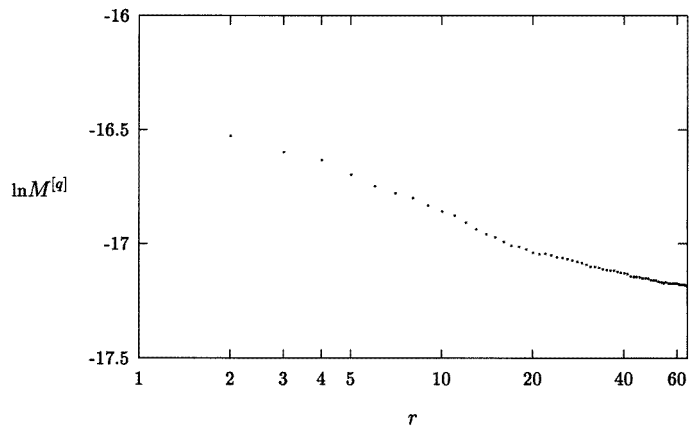

Figure 4. The correlator $M^{[q]}$ (for $q = 0.5$) as a function of the spatial distance r in a log–log plot.

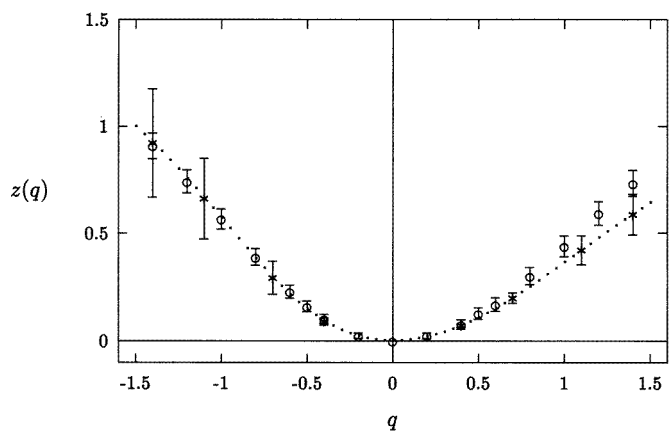

Figure 5. The numerically obtained data for $z(q)$ (*) in comparison with data following from the scaling relation equation (13) (○).

In summary, we have presented numerical evidence for the validity of the scaling relations (12) and (13).

4. Correlations at two different energies

In this section we firstly discuss the scaling relations for correlations of critical eigenstates at two different energies. We stress the role of a length scale L_ω related to the energy separation ω. Secondly, we present numerical calculations which lead to the interpretation of L_ω being a cutoff scale for MP coherence. Thirdly, we verify the validity of scaling relations based on this interpretation. Finally, we discuss the relevance of our results for the interpretation of the local density of states as being an order parameter of the LD transition.

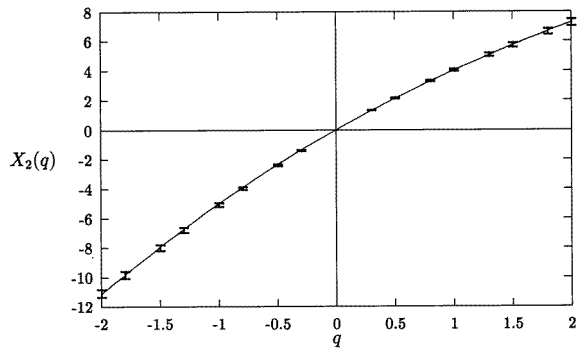

Figure 6. The numerically obtained data for $x_2(q)$. The line shows the data following from the scaling relation equation (12).

We define the q-dependent correlation of box probabilities corresponding to two different eigenstates with energies E and $E+\omega$:

$$M_\omega^{[q]}(r, L_b, L) := \langle [P_i(E; L_b)]^q [P_{i+s}(E+\omega; L_b)]^q \rangle_L. \tag{14}$$

To understand the correlation behaviour of non-localized states with respect to the energy separation one has to compare the relevant energy scales of the problem. These are the average level spacing Δ and the energy $E_c(\omega)$ corresponding to the time a wave packet (formed from states within an energy window of width ω) needs to diffuse through the system, $L^2 = (\hbar/E_c(\omega))D(\omega)$. Here $D(\omega)$ is the corresponding diffusion constant. According to Chalker [9], these scales give rise to the definition of two length scales depending on the energy separation ω:

$$\tilde{L}_\omega := (\omega/E_c(\omega))^{-1/2} L \tag{15}$$

$$L_\omega := (\omega/\Delta)^{-1/d} L. \tag{16}$$

The first length scale, $\tilde{L}_\omega$, is the typical distance a wave packet will travel diffusively in a time $\hbar/\omega$. From this it is natural to assume that correlations between $P_i(E; L_b)$ and $P_{i+s}(E+\omega; L_b)$ will be present at least for distances $r \ll \tilde{L}_\omega$ whereas for larger distances the amplitudes are uncorrelated. Such uncorrelated behaviour is typical for random matrix theory approaches to chaotic systems. The corresponding assumption is known as 'isotropy' or the 'no-preferential-basis' assumption and means that the unitary matrices that diagonalize the Hamiltonian are distributed uniformly in the unitary group and no correlations (apart from the unitarity property) between different matrix elements occur. Thus, the presence of correlations means a breakdown of the 'no-preferential-basis' assumption. In electron systems with spatial disorder, however, a preference to some basis is always given. This preference is lost in $M_\omega^{[q]}$ for distances $r \gg \tilde{L}_\omega$.

The second length scale, L_ω, is the linear size of a system with level spacing ω. Taking the existence of a preferential basis to be significant we will adopt the hypothesis that two wave functions with energetic separation smaller than the level spacing show a spatial correlation behaviour of their amplitudes similar to that corresponding to one of these wave functions, i.e. they are statistically indistinguishable.

At the critical point of the LD transition the conductance becomes a size independent quantity, $G = g^* e^2/h$, and with the help of the Einstein relation between conductivity and diffusion one finds $\tilde{L}_\omega = (g^*)^{1/d} L_\omega$ [9]. Since g^* is of $\mathcal{O}(1)$ the two length scales coincide

at the LD transition. Therefore, we will focus our considerations on the role of L_ω and start with the following working hypothesis:

$M_\omega^{[q]}$ will show the same correlation exponent $z(q)$ as $M_{\omega=0}^{[q]}$ provided $r \ll L_\omega$. For distances $r \gg L_\omega$ the amplitudes $P_i(E; L_b)$ and $P_{i+s}(E+\omega; L_b)$ are uncorrelated.

This means that L_ω is an MP phase coherence length setting a cutoff for correlations. Asking for the scaling properties of $M_\omega^{[q]}$ in the regime $L_b < r < L_\omega \leqslant L$ we make the *ansatz* (cf equation (10))

$$M_\omega^{[q]} \propto L_b^{X_2(q)} r^{-z(q)} L_\omega^{Z(q)} L^{-Y_2(q)}. \tag{17}$$

Here we have already anticipated that the exponent with respect to r is $z(q)$, as given before, and will check whether this is consistent within the following procedure of deriving the scaling relations between $X_2(q)$, $z(q)$, $Z(q)$, $Y_2(q)$ and $\tau(q)$. As in the case of $M^{[q]}$ for a fixed energy we consider limiting situations. In case (i) we keep r of the order of L_b and L_ω of the order of L, whereas in case (ii) we keep r of the order of L_ω. Following our working hypothesis for case (i) we expect no difference from the case with zero energy separation

$$M_\omega^{[q]} \propto \langle [P_i(E; L_b)]^{2q} \rangle_L \tag{18}$$

resulting in

$$X_2(q) - z(q) = d + \tau(2q) = Y_2(q) - Z(q). \tag{19}$$

For case (ii), according to the hypothesis, $M_\omega^{[q]}$ depends neither on r nor on L_ω and approaches the uncorrelated value,

$$M_\omega^{[q]} \propto (\langle [P_i(E; L_b)]^q \rangle_L)^2. \tag{20}$$

This leads to

$$z(q) = Z(q) \qquad X_2(q) = 2d + 2\tau(q) = Y_2(q). \tag{21}$$

Equations (19) and (21) yield the scaling relations

$$X_2(q) = 2d + 2\tau(q) = Y_2(q) \tag{22}$$

$$z(q) = Z(q) = d + 2\tau(q) - \tau(2q) \tag{23}$$

which form a central result of this article. Now the following conclusions can be drawn. (1) The result for $z(q)$ is the same as in the case of zero energy separation. (2) The energy separation ω is not an independent scaling parameter but appears only in the combination L_ω/r. (3) The exponent corresponding to the box size, $x_2(q) = X_2(q)$, is not affected by a finite energy separation but the exponent corresponding to the system size L (which is $y_2(q)$ for zero energy separation) splits up into the exponents $Z(q)$ (corresponding to L_ω) and $Y_2(q)$ (corresponding to L for finite energy separation).

In [5] it has been speculated that for $r \gg L_\omega$ modified power law correlations may still exist. Such behaviour would be in conflict with the interpretation of $\tilde{L}_\omega$ adopted here and it would lead to different scaling relations in comparison to (22) and (23). Especially, the equality between $z(q)$ and $Z(q)$ would be lost for arbitrary values of q. However, the possibility of such behaviour could not be ruled out on the basis of previously obtained numerical data which correspond to the case $q = 1$.

We will now demonstrate that the working hypothesis formulated above and the resulting conclusions (1) and (2) are consistent with our data.

Firstly, we checked whether L_ω serves as a cutoff for correlations. That this is indeed the case can be seen from figure 7 showing the logarithm of $M_\omega^{[q]}$ as a function of $\ln r$ (r is measured in units of l_B) for a fixed size $L_b = l_B$ and $q = 0.5$. The energy separation

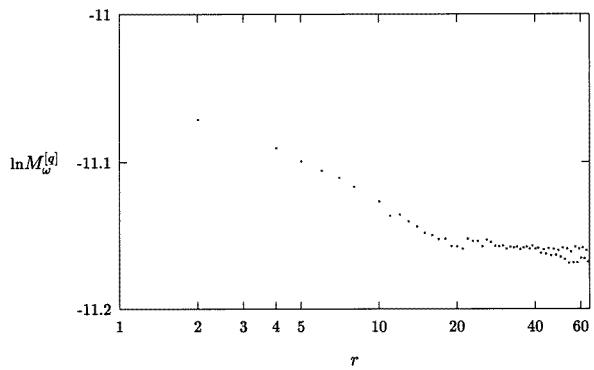

Figure 7. The correlator $M_\omega^{[q]}$ (for $q = 0.5$) as a function of the spatial distance r in a log–log plot for $L_\omega = 30l_B$.

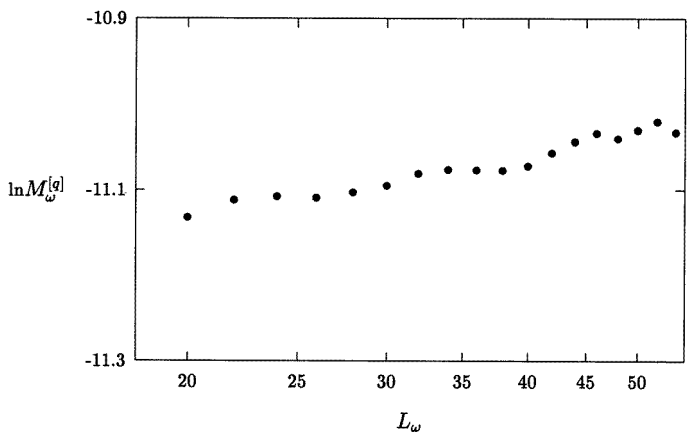

Figure 8. The correlator $M_\omega^{[q]}$ (for $q = 0.5$) as a function of the length scale L_ω in a log–log plot.

ω corresponds to $L_\omega = 30l_B$. A clear r-dependence up to approximately this scale can be observed. In order to calculate a reliable scaling exponent $z(q)$ one has to take $L_\omega \geqslant 100l_B$. Doing so, we found (within the errors) the same $z(q)$ values as for correlations with no energy separation, e.g. $z(1) = 0.39 \pm 0.04$, $z(q = 0.5) = 0.10 \pm 0.03$. Thus, the working hypothesis turned out to be consistent with our data. The next step is to investigate the scaling exponent $Z(q)$ corresponding to L_ω which should, according to the scaling relation equation (23), be equal to $z(q)$, resulting in a combined scaling parameter r/L_ω.

For the numerical determination of $Z(q)$ we used about 130^2 combinations of pairings between different eigenstates. For each pair the actual value of the cutoff scale for correlations fluctuates around the calculated value L_ω. This fact requires a large number of data to extract reliable scaling exponents. From the plots of $\ln M_\omega^{[q]}$ versus $\ln L_\omega$ with fixed values of $r \approx L_b = l_B$ we determined the approximate linear behaviour in a regime between $L_\omega \approx 20l_B$ and $L_\omega \approx 50l_B$. As can be seen in figure 8 ($q = 0.5$) there are fluctuations due to rare pairings. In figure 9 we show the numerically obtained $Z(q)$ function together

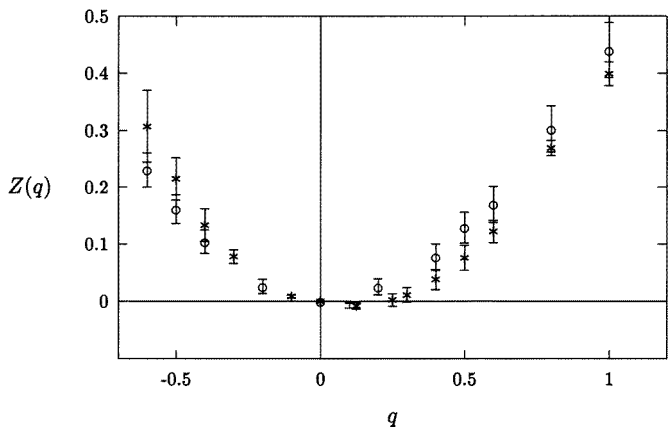

Figure 9. The numerically obtained data for $Z(q)$ (○) in comparison with the numerically obtained data for $z(q)$ (*).

with the errors of the linear fit in log–log plots. It is compared with the previously obtained $z(q)$ function for zero energy separation (cf section 3). The data show that the equality $Z(q) = z(q)$ is consistent with our data, e.g. $Z(1) = 0.38 \pm 0.04$, $Z(0.5) = 0.09 \pm 0.03$ and $\mathrm{d}^2 Z/\mathrm{d}q^2(q = 0) = 1.0 \pm 0.1$.

We have thus demonstrated that the working hypothesis as well as the scaling relations are consistent with our numerical results.

Having established the role of r/L_ω as the relevant scaling parameter for correlations of eigenstate amplitudes with universal exponent $z(q)$ (related to $\tau(q)$ by a scaling relation) let us now discuss the consequences of these findings for the interpretation of the local density of states being an order parameter of the LD transition. The local density of states (LDOS) is defined formally as $\rho(E, \boldsymbol{r}) = \sum_n \delta(E - E_n)|\psi_n(\boldsymbol{r})|^2$ where $\psi_n(\boldsymbol{r})$ are wave functions corresponding to eigenenergies E_n. In a finite system this function has isolated peaks at E_n and becomes a smooth function of energy only for an open system or in the thermodynamic limit. This is also true for the global density of states (DOS) $\rho(E) = L^{-d} \sum_n \delta(E - E_n)$. Since the smallest relevant energy scale for the structure of the DOS as well as of the LDOS is set by the average level spacing Δ a smearing out of the δ-functions over this scale is needed to talk about DOS and LDOS in finite systems. It is known that the average DOS does not reflect the LD transition but is a smooth function of energy and independent of system size L. With this smearing out of the δ-functions we define the LDOS as

$$\rho(E, \boldsymbol{r}) = \Delta(E)^{-1}|\psi(E, \boldsymbol{r})|^2 \tag{24}$$

where $|\psi(E, \boldsymbol{r})|^2$ represents the microcanonical average of squared amplitudes at a given energy E. Since $\Delta(E)$ is smooth in energy and behaves as L^{-d}, the scaling behaviour of the LDOS is determined by that of the wave function. Consequently, we have

$$\langle[\rho(E, \boldsymbol{r})]^q\rangle_L \propto L^{(q-1)d-\tau(q)} \tag{25}$$

and for the typical value

$$\rho_{\mathrm{typ}} = \exp[\langle\ln(\rho(\boldsymbol{r}))\rangle_L] \propto L^{d-\alpha_0} \tag{26}$$

which *does* reflect the LD transition. Scaling L with the localization length $\xi \propto |E - E^*|^{-\nu}$ where E^* and ν are the critical energy and the critical exponent of the localization length,

respectively, we arrive at the conclusion that the typical LDOS vanishes on approaching the critical energy with exponent $\beta_{\rm typ} = \nu(\alpha_0 - d)$. This observation has led to the interpretation of the LDOS being an order parameter field of the LD transition [5] (see also [3, 15]). The unconventional feature as compared to ordinary critical phenomena lies in the facts (i) that the order parameter field has a broad distribution resulting in a non-linear dependence of exponents on the degree of moments considered (multifractality) and (ii) that the average value has vanishing scaling exponent while the typical value gives rise to a positive scaling exponent $\beta_{\rm typ}$. The scaling relations that we derived for the wave function amplitudes transform to scaling relations of the LDOS since each box amplitude has to be multiplied by a constant factor of L^d,

$$\langle(\rho(E, \boldsymbol{r}_1))^q(\rho(E+\omega, \boldsymbol{r}_2))^q\rangle_L \propto (L_\omega/r)^{z(q)} L^{-A(q)} \qquad r = |\boldsymbol{r}_1 - \boldsymbol{r}_2| \tag{27}$$

$$z(q) = d + 2\tau(q) - \tau(2q) \qquad A(q) = 2(1-q)d + 2\tau(q). \tag{28}$$

In this light our scaling relations turn out to be the appropriate scaling relations connecting the spatial correlations of the local order parameter field to its scaling dimensions (cf equations (25), (27) and (28)). However, due to the MP coherence these scaling relations are different from those of ordinary critical exponents where only one correlation exponent appears and no multifractality has to be taken into account. The most remarkable difference lies here in the observation that power law scaling is also present in the MP coherence length $\hat{L}$. The latter is given by the system size L for energy separation smaller than the level spacing or by L_ω for energy separation larger than the level spacing. Our findings demonstrate that the interpretation of the LDOS being an order parameter field for the LD transition is supported by the existence of universal scaling relations. We close this section by pointing out that $z(1) = 2 - \tau(2) \approx 0.4 \neq 0$ (for the correlator of the density of states) with L_ω/r forming the scaling parameter is equivalent (cf [9, 16]) to the phenomenon of 'anomalous diffusion' found by Chalker and Daniell [8]. As pointed out in [8], the anomalous character of diffusion lies in the non-Gaussian dispersion of a wave packet in time t despite the fact that the average diameter grows like $\sqrt{t}$. This non-Gaussian time dispersion is caused by the multifractal character of eigenstates.

5. Conclusions

We have demonstrated explicitly that the distribution of amplitudes of critical eigenstates in quantum Hall systems (QHSs) is contained in the multifractal $f(\alpha)$ spectrum, essentially characterized by one critical exponent, $\alpha_0 \approx 2.28$ (section 2). Following [5] we derived scaling relations which relate the critical exponents of the spatial correlation of amplitudes (for a fixed energy in the critical regime) to the multifractal spectrum (equations (11–13)) and demonstrated that they are consistent with numerical data obtained for QHSs (section 3). Most interesting is the (non-trivial) dependence of the correlator on the system size. In section 4 we considered correlations of amplitudes corresponding to critical eigenstates with energy separation ω. Following Chalker [9] we identified a length scale L_ω (describing the system size with level spacing ω) as the relevant cutoff for correlations. Furthermore, we exploited the hypothesis that the amplitudes are correlated as for zero energy separation provided their distance is much less than L_ω. We found scaling relations (equations (25), (27) and (28)) which relate all the correlation exponents to the universal multifractal spectrum of critical eigenstates. Most important are (i) the confirmation of the scaling parameter L_ω/r and (ii) the identification of L_ω as the upper limit for multifractal correlations. We discussed implications of our results for the statistical properties of the local density of states in the critical regime. Our findings demonstrate that the interpretation of

the LDOS being an order parameter field for the LD transition is supported by the existence of universal scaling relations.

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